

# IRREDUCIBLE SUBQUOTIENTS OF GENERIC GELFAND-TSETLIN MODULES OVER $U_q(\mathfrak{gl}_n)$

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**ABSTRACT.** We provide a classification and explicit bases of tableaux of all irreducible subquotients of generic Gelfand-Tsetlin modules over  $U_q(\mathfrak{gl}_n)$  where  $q \neq \pm 1$ .

## 1. INTRODUCTION

Recently there has been a breakthrough in the theory of Gelfand-Tsetlin modules in the papers [7], [9], [10], [8]. In these papers new classes of simple  $\mathfrak{gl}_n$ -modules were constructed generalising a classical Gelfand-Tsetlin basis [14], [22] for finite-dimensional representations. These new representations also have a basis consisting of Gelfand-Tsetlin tableaux but such tableaux are not necessarily eigenvectors of the Gelfand-Tsetlin subalgebra [5]. This fact requires a modified action of the generators of the Lie algebra on this basis. Gelfand-Tsetlin representations are related to the theory of integrable systems [19], [20], [1], [2], [3], [4], general hypergeometric functions on the complex Lie group  $GL(n)$ , [15],[16]; solutions of the Euler equation, [6], [25] among the others.

The purpose of current paper is to study the Gelfand-Tsetlin basis for quantum  $\mathfrak{gl}_n$  aiming to generalize the constructions above in the quantum case. Previously, partial results were obtained for example in [21], [23], [24], [12]. A general theory of Gelfand-Tsetlin modules for quantum  $\mathfrak{gl}_n$  was developed in [11]. Even though quantization of the Gelfand-Tsetlin basis for generic module in the non-root of unity case may seem straightforward it does require a very careful treatment which was done in this paper. We also include a root of unity case.

Our main result is Theorem 6.2 which provides explicit construction of all irreducible generic Gelfand-Tsetlin modules with tableaux realization. In Section 7 we consider  $q$  a root of unity and apply our construction in this case. It yields new explicit constructions of some finite dimensional irreducible modules.

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## 2. NOTATION AND CONVENTIONS

Throughout the paper we fix an integer  $n \geq 2$ . The ground field will be  $\mathbb{C}$ . For  $a \in \mathbb{Z}$ , we write  $\mathbb{Z}_{\geq a}$  for the set of all integers  $m$  such that  $m \geq a$ . Similarly, we define  $\mathbb{Z}_{< a}$ , etc. By  $U_q$  we denote the quantum enveloping algebra of  $\mathfrak{gl}(n)$ . We

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fix the standard Cartan subalgebra  $\mathfrak{h}$ , the standard triangular decomposition and the corresponding basis of simple roots of  $U_q$ . The weights of  $U_q$  will be written as  $n$ -tuples  $(\lambda_1, \dots, \lambda_n)$ . For a commutative ring  $R$ , by  $\text{Specm } R$  we denote the set of maximal ideals of  $R$ . We will write the vectors in  $\mathbb{C}^{\frac{n(n+1)}{2}}$  in the following form:

$$L = (l_{ij}) = (l_{n1}, \dots, l_{nn} \mid l_{n-1,1}, \dots, l_{n-1,n-1} \mid \dots \mid l_{21}, l_{22} \mid l_{11}).$$

For  $1 \leq j \leq i \leq n$ ,  $\delta^{ij} \in \mathbb{Z}^{\frac{n(n+1)}{2}}$  is defined by  $(\delta^{ij})_{ij} = 1$  and all other  $(\delta^{ij})_{k\ell}$  are zero. For  $i > 0$  by  $S_i$  we denote the  $i$ th symmetric group. Let  $1(q)$  be the set of all complex  $x$  such that  $q^x = 1$ . Finally, for any complex number  $x$ , we set

$$(x)_q = \frac{q^x - 1}{q - 1}, \quad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

### 3. GELFAND-TSETLIN MODULES

Let  $P$  be the free  $\mathbb{Z}$ -lattice of rank  $n$  with the canonical basis  $\{\epsilon_1, \dots, \epsilon_n\}$ , i.e.  $P = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ , endowed with symmetric bilinear form  $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$ . Let  $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} \mid j = 1, 2, \dots\}$  and  $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n-1\}$ . Then  $\Phi$  realizes the root system of type  $A_{n-1}$  with  $\Phi$  a basis of simple roots.

We define  $U_q$  as a unital associative algebra generated by  $e_i, f_i (1 \leq i \leq n)$  and  $q^h (h \in P)$  with the following relations:

$$\begin{aligned} (1) \quad & q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad (h, h' \in P), \\ (2) \quad & q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, \\ (3) \quad & q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i, \\ (4) \quad & e_i f_j - f_j e_i = \delta_{ij} \frac{q^{\alpha_i} - q^{-\alpha_i}}{q - q^{-1}}, \\ (5) \quad & e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \quad (|i - j| = 1), \\ (6) \quad & f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \quad (|i - j| = 1), \\ (7) \quad & e_i e_j = e_j e_i, \quad f_i f_j = f_j f_i \quad (|i - j| > 1). \end{aligned}$$

The quantum special linear algebra  $U_q(sl_n)$  is the subalgebra of  $U_q$  generated by  $e_i, f_i, q^{\pm \alpha_i} (i = 1, 2, \dots, n-1)$ .

**Remark 3.1** ([13], Theorem 12).  $U_q$  has an alternative representation. It is isomorphic to the algebra generated by  $l_{ij}^+, l_{ji}^-, 1 \leq i \leq j \leq n$  subject to the relations

$$\begin{aligned} (8) \quad & RL_1^\pm L_2^\pm = L_2^\pm L_1^\pm R \\ (9) \quad & RL_1^+ L_2^- = L_2^- L_1^+ R \end{aligned}$$

where  $R = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i < j} e_{ij} \otimes e_{ji}$ . The isomorphism between this two representations is given by

$$\begin{aligned} l_{ii}^\pm &= q^{\pm \epsilon_i}, \\ l_{i,i+1}^+ &= (q - q^{-1}) q^{\epsilon_i} e_i, \\ l_{i+1,i}^- &= (q - q^{-1}) f_i q^{\epsilon_i}. \end{aligned}$$

Let for  $m \leq n$ ,  $\mathfrak{gl}_m$  be the Lie subalgebra of  $\mathfrak{gl}(n)$  spanned by  $\{E_{ij} \mid i, j = 1, \dots, m\}$ . We have the following chain

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \dots \subset \mathfrak{gl}_n.$$

It induces the chain  $U_1 \subset U_2 \subset \dots \subset U_n$  for the universal enveloping algebras  $U_m = U(\mathfrak{gl}_m)$ ,  $1 \leq m \leq n$ . If By we denote by  $(U_m)_q$  the quantum universal enveloping algebra of  $\mathfrak{gl}_m$ . We have the following chain  $(U_1)_q \subset (U_2)_q \subset \dots \subset (U_n)_q$ . Let  $Z_m$  denotes the center of  $(U_m)_q$ . The subalgebra of  $U_q$  generated by  $\{Z_m \mid m = 1, \dots, n\}$  will be called the *Gelfand-Tsetlin subalgebra* of  $U_q$  and will be denoted by  $\Gamma_q$ .

**Theorem 3.2** ([13], Theorem 14). *The center of  $U_q(\mathfrak{gl}_m)$  is generated by the following  $m + 1$  elements*

$$c_{mk} = \sum_{\sigma, \sigma' \in S_m} (-q)^{l(\sigma) + l(\sigma')} l_{\sigma(1), \sigma'(1)}^+ \cdots l_{\sigma(k), \sigma'(k)}^+ l_{\sigma(k+1), \sigma'(k+1)}^- \cdots l_{\sigma(m), \sigma'(m)}^-,$$

where  $0 \leq k \leq m$ .

**Definition 3.3.** *A finitely generated  $U$ -module  $M$  is called a Gelfand-Tsetlin module (with respect to  $\Gamma_q$ ) if*

$$(10) \quad M = \bigoplus_{\mathfrak{m} \in \text{Specm } \Gamma_q} M(\mathfrak{m}),$$

where  $M(\mathfrak{m}) = \{v \in M \mid \mathfrak{m}^k v = 0 \text{ for some } k \geq 0\}$ . Equivalently,

$$(11) \quad M = \bigoplus_{\chi \in \Gamma_q^*} M(\chi)$$

where  $M(\chi) = \{v \in M : \forall g \in \Gamma_q, \exists k \in \mathbb{Z}_{>0} \text{ such that } (g - \chi(g))^k v = 0\}$ .

The Gelfand-Tsetlin support of  $M$  is the set  $\text{Supp}_{GT}(M) := \{\chi \in \Gamma_q^* : M(\chi) \neq 0\}$ .

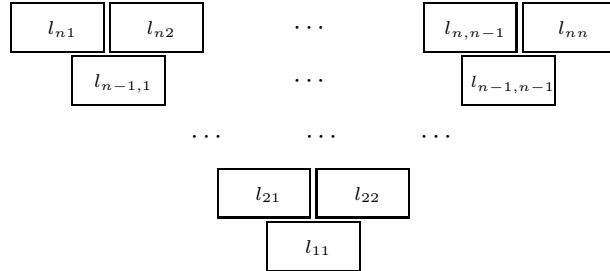
**Lemma 3.4.** *Any submodule of a Gelfand-Tsetlin module over  $U_q$  is a Gelfand-Tsetlin module.*

*Proof.* Analogous to [9] Lemma 3.2. □

#### 4. FINITE DIMENSIONAL MODULES OF $U_q$

In this section we recall the quantum version of a classical result of Gelfand and Tsetlin which provides an explicit basis for every irreducible finite dimensional  $U_q$ -module.

**Definition 4.1.** *For a vector  $L = (l_{ij})$  in  $\mathbb{C}^{\frac{n(n+1)}{2}}$ , by  $T(L)$  we will denote the following array with entries  $\{l_{ij} : 1 \leq j \leq i \leq n\}$*



such an array will be called a Gelfand-Tsetlin tableau of height  $n$ . A Gelfand-Tsetlin tableau of height  $n$  is called standard if  $l_{ki} - l_{k-1,i} \in \mathbb{Z}_{\geq 0}$  and  $l_{k-1,i} - l_{k,i+1} \in \mathbb{Z}_{>0}$  for all  $1 \leq i \leq k \leq n - 1$ .

The following theorem describes the Gelfand-Tsetlin approach for simple finite dimensional  $U_q$  modules with a given highest weight.

**Theorem 4.2** ([24], Theorem 2.11). *Let  $L(\lambda)$  be the finite dimensional irreducible module over  $U_q$  of highest weight  $\lambda = (\lambda_1, \dots, \lambda_n)$ . Then there exist a basis of  $L(\lambda)$  consisting of all standard tableaux  $T(L)$  with fixed top row  $l_{nj} = \lambda_j - j$ . Moreover, the action of the generators of  $U_q$  on  $L(\lambda)$  is given by the Gelfand-Tsetlin formulae:*

$$(12) \quad \begin{aligned} q^{\epsilon_k}(T(L)) &= q^{a_k} T(L), \quad a_k = \sum_{i=1}^k l_{k,i} - \sum_{i=1}^{k-1} l_{k-1,i} + k, \quad k = 1, \dots, n, \\ e_k(T(L)) &= - \sum_{j=1}^k \frac{\prod_i [l_{k+1,i} - l_{k,j}]_q}{\prod_{i \neq j} [l_{k,i} - l_{k,j}]_q} T(L + \delta^{kj}), \\ f_k(T(L)) &= \sum_{j=1}^k \frac{\prod_i [l_{k-1,i} - l_{k,j}]_q}{\prod_{i \neq j} [l_{k,i} - l_{k,j}]_q} T(L - \delta^{kj}). \end{aligned}$$

The next proposition gives the explicit action of the generators of  $\Gamma_q$ .

**Proposition 4.3.** *The generator  $c_{nk}$  of  $\Gamma_q$  acts on  $T(L)$  as a scalar multiplication by*

$$\gamma_{nk}(L) = (k)_{q^{-2}}! (n-k)_{q^{-2}}! q^{k(k+1) + \frac{n(n-3)}{2}} \sum_{\tau} q^{\sum_{i=1}^k l_{n\tau(i)} - \sum_{i=k+1}^n l_{n\tau(i)}}$$

where  $\tau \in S_n$  is such that  $\tau(1) < \dots < \tau(k), \tau(k+1) < \dots < \tau(n)$ .

*Proof.* Analogous to [?] Theorem 5.1. Choose a lowest weight vector  $v = T(L)$  in  $L(\lambda)$ , the entries of  $T(L)$  should satisfy  $l_{ij} = l_{i+1,j+1} + 1$  for any  $i, j$ . Note that the generators  $l_{ij}^+, l_{ji}^-$  belong to the upper and lower Borel subalgebra generated by  $e_i, f_i$  respectively. The element  $l_{\sigma(k+1), \sigma'(k+1)}^- \cdots l_{\sigma(n), \sigma'(n)}^-$  kills  $v$  unless  $\sigma_{k+1} = \sigma'_{k+1}, \dots, \sigma_n = \sigma'_n$ . But  $\sigma_1 \leq \sigma'_1, \dots, \sigma_k \leq \sigma'_k$ , so one must have  $\sigma_i = \sigma'_i$  for all  $1 \leq i \leq n$  in the action of  $c_{nk}$  on  $v$ . We thus have

$$(13) \quad c_{nk}v = \sum_{\sigma \in S_n} q^{2l(\sigma)} q^{a_{\sigma(1)} + \dots + a_{\sigma(k)} - a_{\sigma(k+1)} - \dots - a_{\sigma(n)}} v$$

where

$$\begin{aligned} a_{\sigma(i)} &= \sum_{j=1}^{\sigma(i)} l_{\sigma(i),j} - \sum_{j=1}^{\sigma(i)-1} l_{\sigma(i)-1,j} + \sigma(i) \\ &= l_{\sigma(i),1} + 1 \\ &= \lambda_{n+1-\sigma(i)}. \end{aligned}$$

Then

$$\begin{aligned} \gamma_{nk}(\lambda) &= \sum_{\sigma \in S_n} q^{2l(\sigma)} q^{a_{\sigma(1)} + \dots + a_{\sigma(k)} - a_{\sigma(k+1)} - \dots - a_{\sigma(n)}} \\ &= \sum_{\sigma \in S_n} q^{2l(\sigma)} q^{\lambda_{n+1-\sigma(1)} + \dots + \lambda_{n+1-\sigma(k)} - \lambda_{n+1-\sigma(k+1)} - \dots - \lambda_{n+1-\sigma(n)}} \\ &= \sum_{\sigma \in S_n} q^{n(n-1)-2l(\sigma)} q^{\lambda_{\sigma(1)} + \dots + \lambda_{\sigma(k)} - \lambda_{\sigma(k+1)} - \dots - \lambda_{\sigma(n)}}. \end{aligned}$$

Let  $\tau$  be a permutation in  $S_n$  such that  $\tau(1) < \dots < \tau(k)$ ,  $\tau(k+1) < \dots < \tau(n)$ . One has that

$$\begin{aligned} \gamma_{nk}(\lambda) &= (k)_{q^{-2}}!(n-k)_{q^{-2}}!q^{n(n-1)} \sum_{\tau} q^{-2l(\tau)} q^{\lambda_{\tau(1)} + \dots + \lambda_{\tau(k)} - \lambda_{\tau(k+1)} - \dots - \lambda_{\tau(n)}} \\ &= (k)_{q^{-2}}!(n-k)_{q^{-2}}!q^{n(n-1)} \sum_{\tau} q^{-2 \sum_{i=1}^k (\tau(i)-i) + \sum_{i=1}^k (l_{n\tau(i)} + \tau(i)) - \sum_{i=k+1}^n (l_{n\tau(i)} + \tau(i))} \\ &= (k)_{q^{-2}}!(n-k)_{q^{-2}}!q^{k(k+1) + \frac{n(n-3)}{2}} \sum_{\tau} q^{\sum_{i=1}^k l_{n\tau(i)} - \sum_{i=k+1}^n l_{n\tau(i)}} \end{aligned}$$

□

**Corollary 4.4.** *The generator  $c_{mk}$  of  $\Gamma_q$  acts on  $T(L)$  as multiplication by*

$$(14) \quad \gamma_{mk}(\lambda) = (k)_{q^{-2}}!(m-k)_{q^{-2}}!q^{k(k+1) + \frac{m(m-3)}{2}} \sum_{\tau} q^{\sum_{i=1}^k l_{m\tau(i)} - \sum_{i=k+1}^m l_{m\tau(i)}}$$

where  $\tau \in S_m$  is such that  $\tau(1) < \dots < \tau(k)$ ,  $\tau(k+1) < \dots < \tau(m)$ .

*Proof.* Follows directly from Theorem 4.3 and the fact that eigenvalues of  $\gamma_{nk}$  depend only on the  $n$ -th row of the tableau. □

## 5. GENERIC GELFAND-TSETLIN MODULES OF $U_q$

Recall that  $1(q)$  stands for the set of all complex  $x$  such that  $q^x = 1$ .

**Definition 5.1.** *A Gelfand-Tsetlin tableau  $T(L)$  is called  $q$ -generic if it satisfies the following defining conditions:*

$$l_{ij} - l_{ik} \notin \frac{1(q)}{2} + \mathbb{Z} \text{ for all } 1 \leq i \leq n \text{ and } k \neq j.$$

By  $\mathcal{B}(T(L))$  we will denote the set of all Gelfand-Tsetlin tableaux  $T(R)$  of height  $n$  satisfying  $r_{nj} = l_{nj}$  and  $r_{ij} - l_{ij} \in \mathbb{Z}$  for  $1 \leq j \leq i \leq n-1$ .

**Theorem 5.2** ([21] Theorem 2). *Let  $T(L)$  be a generic tableau, the vector space  $V(T(L)) = \text{span } \mathcal{B}(T(L))$  has a structure of a  $U_q$ -module of finite length with action of the generators of  $U_q$  given by the Gelfand-Tsetlin formulae (12).*

**Proposition 5.3.** *The Gelfand-Tsetlin subalgebra  $\Gamma_q$  separate the tableaux in  $V(T(L))$ . That is, for any two different tableaux in  $V(T(L))$ , there exists an element in  $\Gamma_q$  with different eigenvalues corresponding to the tableaux.*

*Proof.* Let  $T(R)$  and  $T(S)$  be two tableaux with different  $m$ -th row. Assume  $T(R)$  and  $T(S)$  have the same eigenvalue for any element in  $\Gamma_q$ . It is easy to see from (4.3) that  $(q^{2s_{m1}}, \dots, q^{2s_{mm}})$  is a permutation of  $(q^{2r_{m1}}, \dots, q^{2r_{mm}})$ . Therefore, for any  $r_{mi}$ , there exist  $j$  such that  $q^{2r_{mi}} = q^{2s_{mj}}$ , which implies that  $r_{mi} - s_{mj} \in \frac{1(q)}{2}$ . This lead to  $i = j$  and  $r_{mi} = s_{mj}$  which is a contradiction. □

**5.1. Classification of irreducible generic Gelfand-Tsetlin  $U_q$ -modules.** We recall the following result of Mazorchuk and Turowska.

**Theorem 5.4** ([21] Proposition 2). *If  $\mathfrak{n} \in \text{Specm } \Gamma$  is generic, then there exists a unique irreducible Gelfand-Tsetlin module  $N$  such that  $N(\mathfrak{n}) \neq 0$ .*

**Definition 5.5.** *If  $T(R)$  is a  $q$ -generic tableau and  $\mathbf{r} \in \text{Specm} \Gamma_q$  corresponds to  $R$  then, the unique module  $N$  such that  $N(\mathbf{r}) \neq 0$  is called the irreducible Gelfand-Tsetlin module containing  $T(R)$ , or simply, the irreducible module containing  $T(R)$ .*

This section is devoted to an explicit construction of the irreducible Gelfand-Tsetlin module containing  $T(R)$  for every  $q$ -generic tableau  $T(R)$ .

For convenience we introduce and recall some notation.

**Notation 5.6.** *Let  $T(L)$  be a fixed tableau of height  $n$ .*

- (i)  $\mathcal{B}(T(L)) := \{T(L + z) : z \in \mathbb{Z}^{\frac{n(n-1)}{2}}\}$ .
- (ii)  $V(T(L)) = \text{span} \mathcal{B}(T(L))$ .
- (iii) *For any  $T(R) \in \mathcal{B}(T(L))$  and for any  $1 < p \leq n$ ,  $1 \leq s \leq p$  and  $1 \leq u \leq p-1$  we define:*
  - (a)  $\omega_{p,s,u}(T(R)) := r_{p,s} - r_{p-1,u}$ .
  - (b)  $\Omega(T(R)) := \{(p, s, u) : \omega_{p,s,u}(T(R)) \in \frac{1(q)}{2} + \mathbb{Z}\}$
  - (c)  $\Omega^+(T(R)) := \{(p, s, u) : \omega_{p,s,u}(T(R)) \in \frac{1(q)}{2} + \mathbb{Z}_{\geq 0}\}$
  - (d)  $\mathcal{N}(T(R)) := \{T(S) \in \mathcal{B}(T(L)) : \Omega^+(T(R)) \subseteq \Omega^+(T(S))\}$
  - (e)  $W(T(R)) := \text{span} \mathcal{N}(T(R))$
  - (f)  $U_q \cdot T(R)$ : the  $U_q$ -submodule of  $V(T(L))$  generated by  $T(R)$

**5.2. Submodule generated by a single tableau.** In order to find an explicit basis of every irreducible generic module, we first find a basis of  $U_q \cdot T(R)$  for any tableau  $T(R)$  in  $\mathcal{B}(T(L))$ .

**Definition 5.7.** *Given  $T(Q)$  and  $T(R)$  in  $\mathcal{B}(T(L))$ , we write  $T(R) \preceq_{(1)} T(Q)$  if there exist  $g \in \mathfrak{gl}(n)$  such that  $T(Q)$  appears with nonzero coefficient in the decomposition of  $g \cdot T(R)$  into a linear combination of tableaux. For any  $p \geq 1$  we write  $T(R) \preceq_{(p)} T(Q)$  if there exist tableaux  $T(L^{(1)}), \dots, T(L^{(p)})$ , such that*

$$T(R) = T(L^{(0)}) \preceq_{(1)} T(L^{(1)}) \preceq_{(1)} \dots \preceq_{(1)} T(L^{(p)}) = T(Q).$$

As an immediate consequence of the definition of  $\preceq_{(p)}$  we have the following.

**Lemma 5.8.** *If  $T(Q)$ ,  $T(Q^{(0)})$ ,  $T(Q^{(1)})$  and  $T(Q^{(2)})$  are tableaux in  $\mathcal{B}(T(L))$  then:*

- (i)  $T(Q^{(0)}) \preceq_{(p)} T(Q^{(1)})$  and  $T(Q^{(1)}) \preceq_{(q)} T(Q^{(2)})$  imply  $T(Q^{(0)}) \preceq_{(p+q)} T(Q^{(2)})$ .
- (ii)  $T(Q) \preceq_{(1)} T(Q)$ .

The next theorem describes the submodule of  $V(T(L))$  generated by a fixed tableau  $T(R)$ .

**Theorem 5.9.** *Let  $T(L)$  be  $q$ -generic tableau,  $T(R)$  and  $T(S)$  be in  $\mathcal{B}(T(L))$ .*

- (i) *The Gelfand-Tsetlin formulas endow  $W(T(R))$  with a  $U_q$ -module structure.*
- (ii)  *$U_q \cdot T(R) = W(T(R))$ . In particular,  $\mathcal{N}(T(R))$  forms a basis of  $U_q \cdot T(R)$ , and the action of  $U_q$  on  $U_q \cdot T(R)$  is given by the Gelfand-Tsetlin formulas.*
- (iii)  *$U_q \cdot T(R) = U_q \cdot T(S)$  if and only if  $\Omega^+(T(S)) = \Omega^+(T(R))$ .*
- (iv)  *$U_q \cdot T(R) = V(T(L))$  whenever  $\Omega^+(T(R)) = \emptyset$ .*
- (v) *Every submodule of  $V(T(L))$  is finitely generated.*

*Proof.* (i) In order to prove that  $W(T(R))$  is a submodule, it is enough to prove  $U \cdot T(S) \subseteq W(T(R))$  for any  $T(S) \in \mathcal{N}(T(R))$ . We will show  $g \cdot T(S)$  is in  $W(T(R))$  for every (standard) generator of  $U_q$ .

Suppose  $g = e_k$  for some  $1 \leq k \leq n-1$ . By the Gelfand-Tsetlin formulas, we have

$$e_k(T(S)) = - \sum_{j=1}^k \frac{\prod_i [s_{k+1,i} - s_{k,j}]_q}{\prod_{i \neq j} [s_{k,i} - s_{k,j}]_q} T(S + \delta^{kj})$$

If  $e_k(T(S)) \notin W(T(R))$ , then there exist  $k$  and  $j$  such that  $T(S) \in \mathcal{N}(T(R))$  but  $T(S + \delta^{kj}) \notin \mathcal{N}(T(R))$ . That implies

$$\Omega^+(T(R)) \subseteq \Omega^+(T(S)), \text{ and } \Omega^+(T(R)) \not\subseteq \Omega^+(T(S + \delta^{kj})),$$

Hence, there exists  $(p, s, u) \in \Omega^+(T(R))$  such that  $\omega_{p,s,u}(T(S)) \in \frac{1(q)}{2} + \mathbb{Z}_{\geq 0}$  and  $\omega_{p,s,u}(T(S + \delta^{kj})) \notin \frac{1(q)}{2} + \mathbb{Z}_{\geq 0}$ . The latter holds only in two cases:

$$(p, s, u) \in \{(k, j, u), (k+1, s, j) : 1 \leq u \leq k-1; 1 \leq s \leq k+1\}.$$

Note that if neither of these two cases hold, we have  $\omega_{p,s,u}(T(R + \delta^{kj})) = \omega_{p,s,u}(T(S))$ . We consider now each of the two cases separately.

- (a) Suppose  $(p, s, u) = (k, j, u)$ . Then  $\omega_{k,j,u}(T(S)) = s_{kj} - s_{k-1,u} \in \frac{1(q)}{2} + \mathbb{Z}_{\geq 0}$  and  $\omega_{k,j,u}(T(S + \delta^{kj})) = (s_{kj} + 1) - s_{k-1,u} \notin \frac{1(q)}{2} + \mathbb{Z}_{\geq 0}$ , which is impossible.
- (b) Suppose  $(p, s, u) = (k+1, s, j)$ . Then  $\omega_{k+1,s,j}(T(S)) = s_{k+1,s} - s_{ki} \in \frac{1(q)}{2} + \mathbb{Z}_{\geq 0}$  and  $\omega_{k+1,s,i}(T(S + \delta^{ki})) = s_{k+1,s} - (s_{ki} + 1) \notin \frac{1(q)}{2} + \mathbb{Z}_{\geq 0}$ . Hence  $s_{k+1,s} - s_{ki} = 0$  and then the coefficient of  $T(S + \delta^{ki})$  in the decomposition of  $e_k(T(S))$  is

$$-\frac{\prod_i [s_{k+1,i} - s_{k,j}]_q}{\prod_{i \neq j} [s_{k,i} - s_{k,j}]_q} = 0.$$

Therefore, the tableaux that appear with nonzero coefficients in the decomposition of  $e_k(T(S))$  are elements of  $N(T(R))$ . Hence,  $e_k(T(S)) \in W(T(R))$ .

The proof that  $f_k(T(S)) \in W(T(R))$  is analogous to the one of  $e_k(T(S)) \in W(T(R))$ . The case  $q^{\epsilon_k}$  is trivial because  $q^{\epsilon_k}$  acts as a multiplication by a scalar on  $T(S)$  and  $T(S) \in \mathcal{N}(T(R)) \subseteq W(T(R))$ .

(ii) The Gelfand-Tsetlin subalgebra separate tableaux in  $\mathcal{B}(T(L))$ , it is sufficient to prove that for any  $T(S) \in W(T(R))$ ,  $T(S) \preceq_{(p)} T(R)$  for some  $p \in \mathbb{Z}_{>0}$ . Let  $T(S) = T(R + z)$ , we prove the statement by induction on  $\sum_{1 \leq j \leq i < n} |z_{ij}|$ . when  $\sum_{1 \leq j \leq i < n} |z_{ij}| = 1$ , there exist  $z_{ij} = \pm 1$ , all other entries are zero. We consider each case separately.

- (a) Suppose  $z_{ij} = 1$ . Then the coefficient of  $T(S)$  in  $e_i T(R)$  is

$$-\frac{\prod_i [r_{i+1,k} - r_{i,j}]_q}{\prod_{i \neq j} [r_{j,k} - r_{i,j}]_q}.$$

If there exist  $[r_{i+1,k} - r_{i,j}]_q = 0$ , one has  $s_{i+1,k} - s_{i,j} = \frac{1(q)}{2} - 1$ , then  $T(S) \notin W(T(R))$ . Thus  $r_{i+1,k} - r_{i,j} \neq 0$  for any  $k$  which implies  $T(S) \preceq_{(1)} T(R)$

- (b) Suppose  $z_{ij} = -1$ . Similarly the coefficient of  $T(S)$  in  $f_i T(R)$  is not zero.

When  $\sum_{1 \leq j \leq i < n} |z_{ij}| > 1$ , It is sufficient to proof the following statement. Let  $z_{i,j_0}, z_{i+1,j_1}, \dots, z_{i_k,j_k}$  be the nonzero elements such that  $r_{i+t,j_t} - r_{i+t',j_{t'}} \in \frac{1(q)}{2} + \mathbb{Z}$  for any  $1 \leq t, t' \leq k$ , then there exist  $T(S') = T(R + z')$  such that  $\Omega^+ T(S) \subseteq \Omega^+ T(S') \subseteq \Omega^+ T(R)$  and  $|z'_{ij}| \leq |z_{ij}|$ . Let  $t$  be the maximal number such that all the numbers  $z_{i,j_0}, \dots, z_{i+t,j_t}$  have the same sign, then  $\Omega^+(T(R)) \subseteq \Omega^+(T(S - \sum_{s=0}^t \delta^{i+s,j_s})) \subseteq \Omega^+(T(S))$  if the sign is positive.  $\Omega^+(T(R)) \subseteq \Omega^+(T(S +$

$\sum_{s=0}^t \delta^{i+s, j_s}) \subseteq \Omega^+(T(S))$  if the sign is negative. By induction one has that  $T(S) \preceq_{(p)} T(R)$ .

(iii) (iv) and (v) are easy to see from (i) and (ii).  $\square$

## 6. MAIN RESULTS

**Definition 6.1.** For any  $q$ -generic tableau  $T(L)$ , the block associated with  $T(L)$  is the set of all Gelfand-Tsetlin  $U_q$ -modules with Gelfand-Tsetlin support contained in  $\text{Supp}_{GT}(V(T(L)))$ .

Also, for any  $T(R) \in \mathcal{B}(T(L))$ ,  $1 < p \leq n$  and  $1 \leq u \leq p-1$ , define  $d_{pu}(T(R))$  to be the number of distinct elements in  $\{v_{p,s,u}(T(R)) \mid (p,s,u) \in \Omega(T(R))\}$ , where  $\omega_{p,s,u}(T(R)) = u_{p,s,u}(T(R)) + v_{p,s,u}(T(R))$ , with  $u_{p,s,u}(T(R)) \in \frac{1(q)}{2}$  and  $v_{p,s,u}(T(R)) \in \mathbb{Z}$ .

Now we are ready to give the main theorem in the paper.

**Theorem 6.2.** Let  $T(L)$  be  $q$ -generic tableau,  $T(R) \in \mathcal{B}(T(L))$ .

(i) The irreducible module containing  $T(R)$  has a basis of tableaux

$$\mathcal{I}(T(R)) = \{T(S) \in \mathcal{B}(T(R)) : \Omega^+(T(S)) = \Omega^+(T(R))\}.$$

The action of  $U_q$  on this irreducible module is given by the Gelfand-Tsetlin formulas (12).

(ii) The number of irreducible modules in the block associated with  $T(L)$  is:

$$\prod_{1 \leq u \leq p-1 < n} (d_{pu}(T(L)) + 1).$$

In particular,  $V(T(L))$  is irreducible if and only if  $d_{pu}(T(L)) = 0$  for any  $p$  and  $u$ , or equivalently, if and only if  $\Omega(T(L)) = \emptyset$ .

*Proof.* (i) For each tableau  $T(R)$ , we have an explicit construction of the module containing  $T(R)$  (recall Definition 5.5):

$$M(T(R)) := U \cdot T(R) / \left( \sum U \cdot T(S) \right)$$

where the sum is taken over tableaux  $T(Q)$  such that  $\Omega^+(T(R)) \subsetneq \Omega^+(T(S))$  and  $U \cdot T(S)$  is a proper submodule of  $U \cdot T(R)$ . The module  $M(T(R))$  is simple. Indeed, this follows from the fact that for any nonzero tableau  $T(S)$  in  $M(T(R))$  we have  $U \cdot T(S) = U \cdot T(R)$  and, hence,  $T(S)$  generates  $M(T(R))$ . By Theorem 5.9,  $\mathcal{I}(T(R))$  is a basis for  $M(T(R))$ .

(ii) The irreducible modules are in one-to-one correspondence with the subsets of  $\Omega(T(L))$  of the form  $\Omega^+(T(L+z))$ . For any  $T(R) \in \mathcal{B}(T(L))$ , we can decompose  $\Omega(T(R))$  into a disjoint union  $\Omega(T(R)) = \bigsqcup_{p,u} \Omega_{pu}(T(R))$ , where

$$\Omega_{p,u}(T(R)) = \{(p, 1, u), (p, 2, u), \dots, (p, p, u)\} \cap \Omega(T(R)).$$

Now, if  $\Omega_{p,u}^+(T(R)) := \Omega_{p,u}(T(R)) \cap \Omega^+(T(R))$ , one can write  $\Omega^+(T(R)) = \bigsqcup_{p,u} \Omega_{pu}^+(T(R))$ . For  $p, u$  fixed, let us denote by  $s_{p,u}$  the number of different subsets of the form  $\Omega_{p,u}^+(T(R))$ . So, the number of different subsets of the form  $\Omega^+(T(R))$  is  $\prod_{p,u} s_{p,u}$ . It is easy to see that  $s_{pu} = d_{pu}(T(L)) + 1$ .  $\square$



## 7. ROOT OF UNITY CASE

This section is devoted to describing the irreducible module of the quantum enveloping algebra  $U_q$  when the complex parameter  $q$  is a root of unity. In this case denote by  $d$  its order. Since  $q \neq \pm 1$ . We must have  $d > 2$ .

**Theorem 7.1.** [18] *When  $q$  is a root of unity, any irreducible module of  $U_q$  is finite dimensional.*

Denote

$$e = \begin{cases} d & \text{if } d \text{ is odd} \\ d/2 & \text{if } d \text{ is even} \end{cases}$$

It is easy to verify that

$$[x]_q = 0 \iff x = 0 \pmod{e}.$$

**Remark 7.2.** *In the Gelfand-Tsetlin formulae (12), none of the  $[l_{ki} - l_{kj}]_q$  is zero if  $l_{n1} - l_{nn} \leq e$ . So when  $q$  is a root of unity, Theorem 4.2 holds if  $\lambda_1 - \lambda_n \leq e + 1 - n$ . For a generic tableau  $T(L)$  all  $[l_{ki} - l_{kj}]_q$  are not zero. Hence, Theorem 5.2 holds when  $q$  is a root of unity.*

Quantum Gelfand-Tsetlin subalgebra  $\Gamma_q$  separates the tableaux in the following sense.

**Theorem 7.3.** *Let  $q$  be a root of unity,  $T(L)$  a generic tableau. If  $T(R), T(S) \in V(T(L))$  and  $r_{ij} - s_{ij} \neq 0 \pmod{e}, 1 \leq j \leq i < n$ , then  $\Gamma_q$  separates  $T(R)$  and  $T(S)$ .*

*Proof.* Let  $T(R)$  and  $T(S)$  be two tableaux with two different  $m$ -th row. Assume  $T(R)$  and  $T(S)$  have the same eigenvalue for any element in  $\Gamma_q$ . It is easy to see from (4.3) that  $(q^{2s_{m1}}, \dots, q^{2s_{mm}})$  is a permutation of  $(q^{2r_{m1}}, \dots, q^{2r_{mm}})$ . For any  $r_{mi}$ , there exist  $j$  such that  $q^{2r_{mi}} = q^{2s_{mj}}$ . We have that  $r_{mi} - s_{mj} \in \frac{1(q)}{2}$ .  $T(L)$  is  $q$ -generic, one has that  $i = j$ . Since  $r_{ij} - s_{ij} \neq 0 \pmod{e}$ , then  $r_{mi} = s_{mj}$  which is a contradiction.  $\square$

**Proposition 7.4.** *Let  $T(R)$  be a tableau in  $V(T(L))$  and  $N$  the submodule of  $V(T(L))$  generated by  $T(R)$ . If  $g \cdot T(R) = \sum_i c_i T(R_i)$  for some distinct tableaux  $T(R_i)$  in  $\mathcal{B}(T(L))$  and nonzero  $c_i \in \mathbb{C}$ , we have  $T(R_i) \in N$  for all  $i$ .*

*Proof.* Suppose  $g = e_k$  for some  $1 \leq k \leq n - 1$ . By the Gelfand-Tsetlin formulas, we have

$$e_k(T(R)) = - \sum_{j=1}^k \frac{\prod_i [r_{k+1,i} - r_{k,j}]_q}{\prod_{i \neq j} [r_{k,i} - r_{k,j}]_q} T(R + \delta^{kj})$$

Let  $T(R_1)$  and  $T(R_2)$  be any two tableaux in the summation with nonzero coefficients, then  $(r_1)_{ij} - (r_2)_{ij} = 0$  or  $\pm 1$  for any  $1 \leq j \leq i < n$ . It follows from Theorem 7.3 that  $\Gamma_q$  separate these two tableaux. Thus  $T(R_i) \in N$  for all  $i$ .

The proof that  $f_k(T(R)) \in W(T(R))$  is analogous to the one of  $e_k(T(R))$ . The case  $q^{\epsilon_k}$  is trivial because  $q^{\epsilon_k}$  acts as a multiplication by a scalar on  $T(R)$ .  $\square$

### 7.1. Submodule generated by a single tableau.

**Notation 7.5.** *Let  $T(R)$  be a fixed tableau of height  $n$ . We set*

- (a)  $\omega_{p,s,u}(T(R)) := r_{p,s} - r_{p-1,u}$ . If  $\omega_{p,s,u}(T(R)) \in \frac{1(q)}{2} + \mathbb{Z}$ , we denote  $\omega_{p,s,u}(T(R)) = u_{p,s,u}(T(R)) + v_{p,s,u}(T(R))$ , where  $u_{p,s,u}(T(R)) \in \frac{1(q)}{2}$  and  $0 \leq v_{p,s,u}(T(R)) < e$ .
- (b)  $\Omega(T(R)) := \{(p, s, u) : \omega_{p,s,u}(T(R)) \in \frac{1(q)}{2} + \mathbb{Z}\}$
- (c)  $\mathcal{N}(T(R)) := \{T(S) \in \mathcal{B}(T(L)) \mid \omega_{p,s,u}(T(S)) - u_{p,s,u}(T(R)) \in \mathbb{Z}_{\geq 0} \text{ for all } (p, s, u) \in \Omega(T(R))\}$ .
- (d)  $W(T(R)) := \text{span} \mathcal{N}(T(R))$
- (e)  $U_q \cdot T(R)$ : the  $U_q$ -submodule of  $V(T(L))$  generated by  $T(R)$ .

**Theorem 7.6.** *Let  $T(L)$  be a  $q$ -generic Gelfand-Tsetlin tableau,  $T(R)$  and  $T(S)$  be in  $\mathcal{B}(T(L))$ .*

- (i) *The Gelfand-Tsetlin formulas endow  $W(T(R))$  with a  $U_q$ -module structure.*
- (ii)  *$U_q \cdot T(R) = W(T(R))$ . In particular,  $\mathcal{N}(T(R))$  forms a basis of  $U_q \cdot T(R)$ , and the action of  $U_q(\mathfrak{gl}(n))$  on  $U_q \cdot T(R)$  is given by the Gelfand-Tsetlin formulas (12).*
- (iii)  *$U_q \cdot T(R) = U_q \cdot T(S)$  if and only if  $u_{p,s,u}(T(S)) = u_{p,s,u}(T(R))$  for all  $(p, s, u) \in \Omega(T(L))$ .*

*Proof.* (i) In order to prove that  $W(T(R))$  is a submodule, it is enough to show  $g \cdot T(S)$  is in  $W(T(R))$  for every generator of  $U_q$ . The proof is similar to theorem 5.9 (i).

(ii) Similar to theorem 5.9 (ii), it is sufficient to prove that for any  $T(S) \in W(T(R))$ ,  $T(S) \preceq_{(p)} T(R)$  for some  $p \in \mathbb{Z}_{>0}$ .

(iii) It follows from (i) and (ii).  $\square$

**7.2. New constructions of irreducible Gelfand-Tsetlin modules.** In this section we use Gelfand-Tsetlin basis to give a new realisation of some irreducible Gelfand-Tsetlin modules in root of unity case. We assume  $d$  to be odd.

Let  $p = (p_{ij})$ ,  $1 \leq j \leq i < n$  with nonzero entries in  $\mathbb{C}$ ,  $W_{ij}(R)$  be the submodule generated by  $T(R + d\delta^{ij}) - p_{ij}T(R)$ . By Theorem 7.6 a basis for  $W_{ij}(R)$  is the set  $\{T(S + d\delta^{ij}) - p_{ij}T(S) \mid T(S) \in W(T(R))\}$ .

Let  $N = \sum_{\substack{T(R) \in \mathcal{B}(T(L)) \\ 1 \leq j \leq i < n}} W_{ij}(R)$ , and  $M = V(T(L))/N$ .

**Theorem 7.7.**  *$M$  is an irreducible module with dimension  $d^{\frac{n(n-1)}{2}}$ . Moreover,  $M$  has a basis of tableaux  $T(L + m_{ij}\delta^{ij})$ ,  $0 \leq m_{ij} < d$ ,  $1 \leq j \leq i < n$ .*

*Proof.* The submodule  $N$  has a basis  $\{T(R + \delta^{ij}) - p_{ij}T(R) : R \in \mathcal{B}(T(L)), 1 \leq j \leq i < n\}$ . So the subquotient  $M$  has basis  $T(L + m_{ij}\delta^{ij})$ ,  $0 \leq m_{ij} < d$ ,  $1 \leq j \leq i < n$ . We denote the basis of  $M$  by  $I$ . Suppose  $M_1$  is a nonzero submodule of  $M$ , by Proposition (7.3) the basis of  $M_1$  is a subset of  $I$ . From theorem 7.6 and the relations in quotient module  $M$ , one has that  $I \subseteq U_q T(R)$  for any tableau  $T(R)$  in  $I$ . Thus  $M_1 = M$  and  $M$  is irreducible.  $\square$

**Remark 7.8.** *This module is similar to the module constructed in §7.5.5 of [18].*

From now on we will denote by  $\Lambda$  the following set

$$\{(i, j) \mid (i+1, s, j) \in \Omega(T(R)) \text{ for some } 1 \leq s \leq i+1\}.$$

**Definition 7.9.** For any  $T(R) \in \mathcal{B}(T(L))$ ,  $1 < p \leq n$  and  $1 \leq u \leq p-1$ , for  $(i, j) \in \Lambda$  define  $a_{ij}(T(R))$  and  $b_{ij}(T(R))$  as follows

$$\begin{aligned} a_{ij}(T(R)) &= \min\{v_{i+1,s} \mid (i+1, s, j) \in \Omega(T(R))\}, \\ b_{ij}(T(R)) &= \min\{d - v_{i+1,s} \mid (i+1, s, j) \in \Omega(T(R))\}. \end{aligned}$$

Define

$$t_{ij}(T(R)) = \begin{cases} a_{ij}(T(R)) + b_{ij}(T(R)) & \text{for } (i, j) \in \Lambda \\ d & \text{for } (i, j) \notin \Lambda. \end{cases}$$

**Definition 7.10.** Let  $\Lambda_1$  be a subset of  $\Lambda$ ,  $\Lambda_2 = \Lambda \setminus \Lambda_1$ ,  $\widetilde{M}(T(R))$  be quotient of  $U_q \cdot T(R)$  by

$$\left( \sum_{(i,j) \notin \Lambda} W_{ij}(R) + \sum_{T(S_1)} U_q(T(S_1)) + \sum_{T(S_2)} U_q(T(S'_2) - p_{ij}T(S_2)) \right),$$

where  $T(S_t)$ ,  $t = 1, 2$  run through over the set of tableaux in  $\mathcal{N}(T(R))$  such that  $(i, j) \in \Lambda_t$ ,  $\omega_{i-1,s,j}T(S'_2) - \omega_{i-1,s,j}T(S_2) = d$  for some  $(i-1, s, j) \in \Omega(T(R))$ ,  $\omega_{p,s,u}T(S_2) - \omega_{p,s,u}T(S'_2) = 0$  for any  $(p, s, u) \neq (i-1, s, j)$ .

**Theorem 7.11.**  $\widetilde{M}(T(R))$  is an irreducible module of dimension  $\prod_{1 \leq j \leq i < n} t_{ij}(T(R))$ .

*Proof.* The subquotient  $U_q T(R) / \sum_{T(S)} U_q(T(S))$  has basis

$$I = \{T(S) \mid u_{p,s,u}(T(S)) = u_{p,s,u}(T(R)) \text{ for all } (p, s, u) \in \Omega(T(L))\}.$$

The module  $\widetilde{M}(T(R))$  can be regarded as the subquotient of  $U_q T(R) / \sum_{T(S)} U_q(T(S))$ . Then it has basis:  $\{T(S) \in I \mid s_{ij} = r_{ij} + m_{ij}, 0 \leq m_{ij} < d, (i, j) \notin \Lambda\}$ . Similar to theorem 7.7  $\widetilde{M}(T(R))$  is irreducible. For any  $(i, j) \in \Lambda$ , if we fix the  $i+1$ -th row of the tableau, the number of distinct  $s_{ij}$  in  $I$  is  $t_{ij}(T(R))$ . For  $(i, j) \notin \Lambda$ , there are  $d$  different  $s_{ij}$ . Thus the dimension of  $\widetilde{M}(T(R))$  is  $\prod_{1 \leq j \leq i < n} t_{ij}(T(R))$ .  $\square$

**7.3. Example.** Recall two families  $d$ -dimensional modules of  $U_q(sl_2)$  [17]. The first depends on three complex numbers  $\lambda, a$  and  $b$ . We assume  $\lambda \neq 0$ . Consider the  $d$ -dimensional vector space with a basis  $\{v_0, v_1, \dots, v_{d-1}\}$ . for  $0 \leq p \leq d-1$ , set

$$(15) \quad Kv_p = \lambda q^{-2p} v_p,$$

$$(16) \quad Ev_{p+1} = \left( \frac{q^{-p}\lambda - q^p\lambda^{-1}}{q - q^{-1}} [p+1]_q + ab \right) v_p,$$

$$(17) \quad Fv_p = v_{p+1},$$

and  $Ev_0 = av_{d-1}$ ,  $Fv_{d-1} = bv_0$ , and  $Kv_{d-1} = \lambda q^{-2(d-1)} v_p$ . These formula endow the vector space with a  $U_q$ -module structure, denoted by  $V(\lambda, a, b)$ .

The second family depends on two scalars  $\mu \neq 0$  and  $c$ . Let  $E, F, K$  act on the vector space with basis  $\{v_0, v_1, \dots, v_{d-1}\}$  by

$$(18) \quad Kv_p = \mu q^{2p} v_p,$$

$$(19) \quad Fv_{p+1} = \frac{q^{-p}\mu^{-1} - q^p\mu}{q - q^{-1}} [p+1]_q v_p,$$

$$(20) \quad Ev_p = v_{p+1},$$

and  $Fv_0 = 0, Ev_{d-1} = cv_0$ , and  $Kv_{e-1} = \mu q^{-2}v_{e-1}$ . These formula endow the vector space with a  $U_q$ -module structure, denoted by  $\tilde{V}(\mu, c)$ .

**Theorem 7.12.** [17] *Any irreducible  $U_q$  module of dimension  $d$  is isomorphic to one of the following list:*

- (i)  $V(\lambda, a, b)$  with  $b \neq 0$ ,
- (ii)  $V(\lambda, a, 0)$  where  $\lambda$  is not of the form  $\pm q^{j-1}$  for any  $1 \leq j \leq d-1$ ,
- (iii)  $\tilde{V}(\pm q^{1-j}, c)$  with  $c \neq 0$  and  $1 \leq j \leq d-1$ .

In the following we will compare above modules with modules in 7.7 and 7.11. Let  $x, y, z$  be three complex number,  $v_p = (x, y|z-p), 0 \leq p \leq d-1$ . Consider the vector space with basis of tableaux  $\{T(v_p) : 0 \leq p \leq d-1\}$ . Theorem 7.7 endows the vector space with a  $U_q$ -module structure. The actions of  $E, F, K$  are given by

$$(21) \quad KT(v_p) = q^{2z-(x+y+1)}q^{-2p}T(v_p),$$

$$(22) \quad ET(v_{p+1}) = -[x+p+1-z]_q[y+p+1-z]_qT(v_p),$$

$$(23) \quad FT(v_p) = T(v_{p+1}),$$

and  $ET(v_0) = -s[x-z]_q[y-z]_qT(v_{d-1}), FT(v_{d-1}) = \frac{1}{s}T(v_0)$ . Let  $\lambda = q^{2z-(x+y+1)}, b = \frac{1}{s}, a = -s[x-z]_q[y-z]_qv_{d-1}$ , this module is isomorphic to  $V(\lambda, a, b)$  with  $b \neq 0$ .

Let  $x, y, z$  be three complex number with  $x-z$  or  $y-z \in \frac{1(q)}{2}$ . Consider be the vector space with basis of tableaux  $\{T(v_p) : 0 \leq p \leq d-1\}$ , where  $v_p = (x, y|z-p), 0 \leq p \leq d-1$ . Theorem 7.11 endows the vector space with a  $U_q$ -module structure. The actions of  $E, F, K$  are given by

$$(24) \quad KT(v_p) = q^{2z-(x+y+1)}q^{-2p}T(v_p),$$

$$(25) \quad ET(v_{p+1}) = -[x+p+1-z]_q[y+p+1-z]_qT(v_p),$$

$$(26) \quad FT(v_p) = T(v_{p+1}),$$

and  $Ev_0 = 0, Fv_{d-1} = sv_0$ . This module is isomorphic to  $V(\lambda, 0, s), \lambda = q^{2z-(x+y+1)}$ .

There exist an algebra endomorphism of  $U_q(sl_2)$  such that  $E \mapsto F, F \mapsto E, K \mapsto K^{-1}$ .  $V(\lambda, a, 0)$  and  $\tilde{V}(\mu, c)$  can be obtained from  $V(\lambda, 0, b)$  by the algebra endomorphism.

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